Abstract—Modes in square resonators are analyzed and classified according to the irreducible representations of the point group \(C_{4v}\). If the mode numbers \(p\) and \(q\) that denote the number of wave nodes in the directions of two orthogonal square sides are unequal and have the same even–odd characteristics, the corresponding double modes are accidentally degenerate and can be combined into two new distributions with definite parities relative to the square diagonal mirror planes. The distributions with odd parities belong to the whispering-gallery-like modes in square resonators. The mode frequencies and quality factors are also calculated by the finite-difference time-domain technique and Padé approximation method. The numerically calculated mode frequencies agree with the theoretical ones very well and the whispering-gallery-like modes have quality factors much higher than other modes, including their accidentally degenerate counterparts in square resonators.

Index Terms—Microcavities, square resonators, whispering-gallery (WG) modes, semiconductor lasers.

I. INTRODUCTION

MICROOPTICAL resonators have attracted great interest due to their applications in microcavity lasers and optical add-drop filters. As typical representatives, microdisk and microring resonators are successfully used in the fabrication of microlasers [1]–[4] and add-drop filters [5]–[7]. However, doubly degenerate whispering-gallery (WG) modes exist in the microdisk optical resonator, and real single-mode operation is difficult to realize in the microdisk lasers. Etching gratings with predetermined periods along the microdisk perimeter are used to eliminate the degeneracy [4]. Recently, polygon microcavities have also attracted interest in the microcavity lasers and add-drop filters [8]–[11]. Square resonators based on square-shaped silica fibers are used as filters, and high coupling efficiency is obtained by coupling a prism to the WG-like modes in the square resonator [9], [10]. We have analyzed mode characteristics of the WG-like modes in the square resonator by assuming the diagonal lines to be zero lines for the electric and magnetic fields of the TM and TE modes, respectively [12]. The WG-like modes are modes with the highest quality factors in the square resonator, but a lot of modes predicted by the finite-difference time-domain (FDTD) numerical simulation technique cannot be attributed to the WG-like modes [12].

In this paper, we analyze mode frequencies and field distributions for all modes confined in two-dimensional (2-D) square resonators based on Marcatili’s scheme [13]. We find that the modes can form all the irreducible representations of the point group \(C_{4v}\), of which those that form the \(A_2\) and \(B_1\), irreducible representations are just the WG-like modes. By using the FDTD technique [14] and Padé approximation method [15], we also calculate the mode frequencies and quality factors and mode field distributions numerically and find that the mode frequencies and field distributions are in good agreement with the analytical results. The accidentally degenerate counterparts of the WG-like modes have even parities relative to the square diagonals, and have quality factors of one order of magnitude less than the WG-like modes.

II. MODE FREQUENCIES AND DISTRIBUTIONS

A 2-D square resonator with side length \(a\) is schematically shown in Fig. 1, where \(\Omega_1\) is the inner region of the square resonator with refractive index \(n_1\) and the cladding regions \(\Omega_i(\bar{i} = 2–5)\) and the corner regions (shown shaded on Fig. 1) with air. Similar to Marcatili’s scheme [13] that was used in the analysis of rectangle dielectric waveguides, for confined modes in square resonators we can approximately assume that the field penetration into the cladding regions is exponentially decayed as moving away from the square sides and the field penetration into the four corner regions is so little that it can be ignored. Thus, the magnetic field \(H_z\) for the TM mode or the electric field \(E_z\) for the TM mode in the square resonator can be expressed as

\[
P_{pq}^{t} = \begin{cases} 
\cos(\kappa_{px} x - \varphi_x) \cos(\kappa_{py} y - \varphi_y); & \text{in } \Omega_1 \\
\cos(\kappa_{py} y - \varphi_y) \cos(\frac{\kappa_{px} a}{2} - \varphi_x) \exp[-\gamma_x (x - \frac{a}{2})]; & \text{in } \Omega_2 \\
\cos(\kappa_{px} x - \varphi_x) \cos(\frac{\kappa_{py} a}{2} - \varphi_y) \exp[-\gamma_y (y - \frac{a}{2})]; & \text{in } \Omega_3 \\
\cos(\kappa_{py} y - \varphi_y) \cos(-\frac{\kappa_{px} a}{2} - \varphi_x) \exp[\gamma_x (x + \frac{a}{2})]; & \text{in } \Omega_4 \\
\cos(\kappa_{px} x - \varphi_x) \cos(-\frac{\kappa_{py} a}{2} - \varphi_y) \exp[\gamma_y (y + \frac{a}{2})]; & \text{in } \Omega_5 
\end{cases}
\]

where \(p\) and \(q\) are the mode numbers which denote the number of wave nodes in the \(x\) and \(y\) directions, respectively, \(\varphi_{\nu}(\nu = x, y)\) is zero or \(\pi/2\) when the mode numbers \(p\) and \(q\) are even or odd, and \(\kappa_{\nu}\) and \(\gamma_{\nu}\) are the propagation and decay constants satisfying the following relations:

\[
\kappa_{px}^2 + \kappa_{py}^2 = \kappa_{01}^2 \\
\gamma_{px}^2 + \gamma_{py}^2 = (\frac{n_2^2}{1}) k_0^2
\]

where \(\nu = x, y\) and \(k_0 = ((2\pi)/\lambda)\) is the wavenumber in vacuum. Based on the continuous condition of the tangential electric field for the TE mode or the tangential magnetic field.
The operators of the point group $C_{4v}$ as well as the coordinates are indicated.

for the TM mode at the square sides, we obtain the mode eigen-equations as

$$\kappa_x \tan\left(\frac{\kappa_x a}{2} - \psi_x\right) = n \gamma_x$$

where $\nu = x, y$, and $\eta = n_2^2$ and 1 for the TE and TM modes, respectively. For guided modes, their incident angles at the sides should be larger than the critical angle of total internal reflection between the core and the cladding mediums, which produces the limitations for the modes in the square resonator as [10]

$$\tan^{-1}\left(\frac{\kappa_x}{\kappa_y}\right) > \sin^{-1}\left(\frac{1}{n_1}\right)$$

Combining (2)–(6), we can calculate the confined mode frequencies. In Table I, we list the analytical TE mode frequencies limited in the range of 160–220 THz for a square resonator with a side length of 2 $\mu$m and refractive index 3.2.

### III. Mode Symmetries

The symmetry of a square can be described by the point group $C_{4v}$, whose characters are listed in Table II and corresponding operators are shown in Fig. 1 [16].

The field distribution given in (1) can be classified into the irreducible representations of the point group $C_{4v}$. If the mode number $p$ is even and equal to $q$, then $F_{x}^{\nu p}$ forms the $A_{1}$ representation of the point group $C_{4v}$. If $p$ is odd and equal to $q$, then $F_{x}^{\nu p}$ forms the $B_{2}$ representation. If $p$ is unequal to $q$, there are two degenerate modes $F_{x}^{\nu q}$ and $F_{x}^{\nu p}$. If $p$ is even and $q$ is odd or reversal, the doubly degenerate modes $F_{x}^{\nu q}$ and $F_{x}^{\nu p}$ form the $E$ representation. However, if both $p$ and $q$ are even or odd, the double degeneracies are accidental, i.e., they can be reduced into two one-dimensional (1-D) irreducible representations. $F_{x}^{\nu q}$ and $F_{x}^{\nu p}$ can be combined into double new mode distributions as

$$f_{x}^{\nu q}(p,q) = F_{x}^{\nu q} + F_{x}^{\nu p}$$

$$f_{x}^{\nu p}(p,q) = F_{x}^{\nu q} - F_{x}^{\nu p}.$$  

The new distributions have definite parities relative to the square diagonal mirror planes. The superscripts “$c$” and “$d$” indicate even or odd parities relative to the operator $\sigma_{dl}$ shown in Fig. 1.

<table>
<thead>
<tr>
<th>Analytical</th>
<th>FDTD and Padé</th>
<th>FDTD and Padé</th>
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<tbody>
<tr>
<td>Frequency (THz)</td>
<td>Frequency (THz)</td>
<td>Q-factor</td>
</tr>
<tr>
<td>$H_{x}^{2,6}$</td>
<td>173.55</td>
<td>171.86</td>
</tr>
<tr>
<td>$H_{x}^{2,7}$</td>
<td>193.49</td>
<td>*</td>
</tr>
<tr>
<td>$H_{x}^{2,8}$</td>
<td>211.94</td>
<td>*</td>
</tr>
<tr>
<td>$h_{x}^{0,(3,5)}$</td>
<td>166.27</td>
<td>165.34</td>
</tr>
<tr>
<td>$H_{x}^{3,6}$</td>
<td>185.75</td>
<td>184.48</td>
</tr>
<tr>
<td>$h_{x}^{0,(0,7)}$</td>
<td>205.80</td>
<td>204.44</td>
</tr>
<tr>
<td>$H_{x}^{4,4}$</td>
<td>163.44</td>
<td>162.56</td>
</tr>
<tr>
<td>$H_{x}^{4,5}$</td>
<td>180.68</td>
<td>179.56</td>
</tr>
<tr>
<td>$h_{x}^{0,(4,6)}$</td>
<td>199.02</td>
<td>197.90</td>
</tr>
<tr>
<td>$H_{x}^{4,7}$</td>
<td>218.19</td>
<td>216.87</td>
</tr>
<tr>
<td>$H_{x}^{5,5}$</td>
<td>196.58</td>
<td>195.24</td>
</tr>
<tr>
<td>$H_{x}^{5,6}$</td>
<td>213.73</td>
<td>212.38</td>
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</tbody>
</table>

<table>
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<tr>
<th>Character Table of the Point Group $C_{4v}$</th>
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<tbody>
<tr>
<td>$C_{4v}$</td>
</tr>
<tr>
<td>$A_{1}$</td>
</tr>
<tr>
<td>$A_{2}$</td>
</tr>
<tr>
<td>$B_{1}$</td>
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<tr>
<td>$B_{2}$</td>
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<tr>
<td>$E$</td>
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</tbody>
</table>

If both $p$ and $q$ are even, $f_{x}^{c}$ forms the $A_{1}$ representation and $f_{x}^{d}$ forms the $H_{1}$ representation. If both $p$ and $q$ are odd, $f_{x}^{c}$ forms the $B_{2}$ representation and $f_{x}^{d}$ forms the $A_{2}$ representation. For the square resonator with a side length of 2 $\mu$m and refractive index of 3.2, we plot the $h_{x}^{c(3,5)}$ and $h_{x}^{d(3,5)}$ mode distributions in Fig. 2(a)–(b), $h_{x}^{c(4,6)}$ and $h_{x}^{d(4,6)}$ mode distributions in Fig. 2(c)–(d), and $h_{x}^{c(3,7)}$ and $h_{x}^{d(3,7)}$ mode distributions in Fig. 2(e)–(f) for the TE modes. By manipulating these figures according to the operators of the group $C_{4v}$, we can prove that
IV. THE WHISPERING-GALLERY-LIKE MODES

According to the above description, there are two accidentally degenerate modes if the mode numbers \( p \) and \( q \) are unequal and have the same even–odd characteristics. The double modes have definite parities relative to the square diagonal mirror planes: even or odd. In [12], the modes with odd parities relative to the square diagonals were described as the WG-like modes and marked by the transverse mode number \( m \) and longitudinal mode number \( l \). These mode numbers are found to have the following relations with the mode numbers \( p \) and \( q \) used in this paper:

\[
\begin{align*}
  m &= \frac{|p - q|}{2} - 1 \\
  l &= \begin{cases} 
  p + q + 4, & \text{even } m \\
  p + q + 2, & \text{odd } m.
\end{cases}
\end{align*}
\]

If \(|p - q| = 2\), the corresponding modes form the WG-like modes with fundamental transverse mode distributions. If \(|p - q| = 4\), the corresponding modes form the WG-like modes with first-order transverse mode distributions, and so on. Based on the analytical formulas in [12], we also plot the field distributions of the WG-like modes in the square resonator with a side length of 2 \( \mu \)m and refractive index 3.2. In Fig. 3(a)–(c) for the TE modes, the field distributions coincide with the distributions of \( h_2^x(3,5) \), \( h_2^x(4,6) \), and \( h_2^x(3,7) \) in Fig. 2(b), (d), and (f) very well. The frequencies of these modes given by the formulas in [12] are 166.75, 199.35, and 207.43 THz, respectively, which have larger offsets with the FDTD results than the frequencies obtained by the Marcatili’s scheme, especially for the first-order transverse modes. In fact, less approximation is used in this paper than in [12].

V. NUMERICAL SIMULATION

We also calculate the mode frequencies and quality factors by the FDTD technique [14] for TE modes in the square resonator with a side length of 2 \( \mu \)m and refractive index of 3.2. A uniform mesh with cell size of 10 nm and a 20-cell perfectly matched layer (PML) absorbing boundary condition [17] are used in the FDTD simulation. Impulses of the Gaussian modulated cosine functions are added to points inside the square with low symmetries to excite all possible modes spatially and spectrally. We record the time transience of the magnetic field at points inside the square and then use the Padé approximation method [15] to calculate the intensity spectra. In order to distinguish degenerate modes with the same frequencies, the distribution symmetries are exploited in the FDTD simulation. For example, the distributions of \( h_2^x \) and \( h_2^y \) have even and odd parities relative to the square diagonals, which can be used to distinguish them in the FDTD simulation. A Lorentzian line-shape function is used to fit the intensity spectra with the frequencies \( f_{\text{max}} \) corresponding to the local maxima and the full-width at half-maximum (FWHM) \( \Delta f \) determined from the fitting process. Then the quality factor can be calculated as

\[
Q = \frac{f_{\text{max}}}{\Delta f}.
\]

The intensity spectrum obtained by the FDTD technique and Padé approximation is plotted in Fig. 4, and all the peaks are
marked by the mode numbers \( p \) and \( q \) as \((p,q)\). The calculated mode frequencies and quality factors are listed in Table I. The numerically calculated mode frequencies are in good agreement with the analytical ones. However, for the analytically predicted modes \( H_{2}^{2.7} \) and \( H_{2}^{2.8} \), we do not observe evident peaks corresponding to them in the intensity spectra, which means that their losses are large and their quality factors are consequently low. It is also found that the WG-like modes have quality factors much higher than other resonant modes in the square resonator including their accidentally degenerate counterparts. For instance, the \( h_{2}^{c(3,5)} \) mode has a quality factor of 130 and the \( h_{2}^{c(4,0)} \) mode has a quality factor of 110, which are one order of magnitude less than the \( h_{2}^{c(3,5)} \) and \( h_{2}^{c(4,0)} \) modes. By compressing the impulse spectral width narrow enough to ensure just one mode excited, we can obtain the mode distributions from the FDTD technique. The numerically calculated field distributions of the WG-like \((0, 12), (0, 14), \) and \((1, 12)\) modes are shown in Fig. 3(d)–(f), which agree with the analytical field distributions given in Fig. 2(b), (d), and (f) and Fig. 3(a)–(c) very well.

VI. CONCLUSION

Modes in square resonators are analyzed with mode frequencies and distributions calculated. The modes are classified into the irreducible representations of the point group \( C_{4v} \), that describes the symmetry of a square. If the mode numbers \( p \) and \( q \) that denote the number of wave nodes in the \( x \) and \( y \) directions are unequal and both even or odd, the corresponding double modes are accidentally degenerate and can be combined into two new distributions which have definite parities relative to the square diagonal mirror planes. The distributions that have odd parities are just the WG-like modes in square resonators. The FDTD technique and Padé approximation method are also used to calculate mode frequencies and quality factors in square resonators. The calculated mode frequencies agree with the theoretical results very well and the quality factors of the WG-like modes are much higher than other modes. The characteristics of the WG-like modes in square resonators make them suitable for fabricating low-threshold semiconductor microcavity lasers with real single-mode operation.

REFERENCES


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